DOI: 10.2478/auom-2020-0005
An. Şt. Univ. Ovidius Constanţa **Vol. 28(1),2020, 81–94**

On Mahler's p-adic S-, T-, and U-numbers

Yann Bugeaud and Gülcan Kekeç

Abstract

We consider some lacunary power series with rational coefficients in $\mathbb{Q}_p$. We show that under certain conditions these series take transcendental values at non-zero rational number arguments, and we determine the classes of these transcendental values with respect to Mahler's classification of p-adic numbers.

1 Introduction

Throughout the present paper, p denotes a fixed prime number, and $|\cdot|_p$ denotes the p-adic absolute value on the field $\mathbb{Q}$ of rational numbers, normalised such that $|p|_p = p^{-1}$. We denote the unique extension of $|\cdot|_p$ to the field $\mathbb{Q}_p$ of p-adic numbers by the same notation $|\cdot|_p$.

In 1955, Roth [11] proved that irrational real numbers which can be approximable by rational numbers at an order greater than 2 are transcendental.

Theorem 1.1 (Roth [11], 1955)**.** *Let ξ be a real number and ε be a positive real number. Suppose that there exists a sequence $(p_n/q_n)_{n=1}^{\infty}$ of rational numbers such that $2 \le q_1 < q_2 < \cdots$ and*

$$0 < \left|\xi - \frac{p_n}{q_n}\right| < q_n^{-2-\varepsilon} \qquad (n = 1, 2, 3, \ldots).$$

Key Words: Mahler's classification of p-adic numbers, p-adic S-number, p-adic T-number, p-adic U-number, transcendence measure.
2010 Mathematics Subject Classification: Primary 11J61; Secondary 11J82.
Received: 16.03.2019
Revised: 10.04.2019
Accepted: 15.07.2019

Then ξ is transcendental.

In 1958, Ridout [10] proved the p-adic analogue of Theorem 1.1. We denote by $|x, y|$ the maximum of $|x|$ and $|y|$, where x and y are non-zero integers with $\gcd(x, y) = 1$.

Theorem 1.2 (Ridout [10], 1958)**.** *Let ξ be a p-adic number and ε be a positive real number. Suppose that there exists a sequence $(x_n/y_n)_{n=1}^{\infty}$ of rational numbers with* $\gcd(x_n, y_n) = 1$ $(n = 1, 2, 3, \ldots)$ *such that* $2 \le |x_1, y_1| < |x_2, y_2| < \cdots$ *and*

$$0 < \left|\xi - \frac{x_n}{y_n}\right|_p < |x_n, y_n|^{-2-\varepsilon} \qquad (n = 1, 2, 3, \ldots).$$

Then ξ is transcendental.

In 1932, Mahler [5] introduced a classification of real transcendental numbers. He divided real transcendental numbers into three disjoint classes and called the numbers in these classes S-, T-, and U-numbers. Later, in 1935, Mahler [6] proposed a classification of p-adic transcendental numbers in analogy with his classification of real transcendental numbers. He divided p-adic transcendental numbers into three disjoint classes and called the numbers in these classes p-adic S-, T-, and U-numbers. (See Bugeaud [3] for information about Mahler's classification in $\mathbb{R}$ and in $\mathbb{Q}_p$.)

In 1964, by adding an assumption on the growth of the sequence $(q_n)_{n=1}^{\infty}$ in Theorem 1.1, Baker [2] established a more precise conclusion than the simple transcendence of ξ.

Theorem 1.3 (Baker [2], 1964)**.** *Let ξ be a real number and ε be a positive real number. Suppose that there exists a sequence $(p_n/q_n)_{n=1}^{\infty}$ of rational numbers with* $\gcd(p_n, q_n) = 1$ $(n = 1, 2, 3, \ldots)$ *such that* $2 \le q_1 < q_2 < \cdots$ *and*

$$0 < \left|\xi - \frac{p_n}{q_n}\right| < q_n^{-2-\varepsilon} \qquad (n = 1, 2, 3, \ldots).$$

If

$$\limsup_{n\to\infty} \frac{\log q_{n+1}}{\log q_n} < \infty,$$

then ξ is either an S-number or a T-number.

Recently, in 2018, Bugeaud and Kekeç [4, Theorem 1.4] proved the p-adic analogue of Theorem 1.3 by following the method of the new proof of Theorem 1.3 introduced in [1, Théorème 3.1].

Theorem 1.4 (Bugeaud and Kekeç [4], 2018)**.** *Let ξ be a p-adic number and ε be a positive real number. Suppose that there exists a sequence $(x_n/y_n)_{n=1}^{\infty}$ of rational numbers with* $\gcd(x_n, y_n) = 1$ $(n = 1, 2, 3, \ldots)$ *such that* $2 \leq |x_1, y_1| < |x_2, y_2| < \cdots$ *and*

$$0 < \left|\xi - \frac{x_n}{y_n}\right|_p < |x_n, y_n|^{-2-\varepsilon} \qquad (n = 1, 2, 3, \ldots).$$

If

$$\limsup_{n\to\infty} \frac{\log |x_{n+1}, y_{n+1}|}{\log |x_n, y_n|} < \infty,$$

then ξ is either a p-adic S-number or a p-adic T-number. Suppose that there exists a sequence $(x_n)_{n=1}^{\infty}$ of integers such that $2 \leq |x_1| < |x_2| < \cdots$ and

$$0 < |\xi - x_n|_p < |x_n|^{-1-\varepsilon} \qquad (n = 1, 2, 3, \ldots).$$

If

$$\limsup_{n\to\infty} \frac{\log |x_{n+1}|}{\log |x_n|} < \infty,$$

then ξ is either a p-adic S-number or a p-adic T-number.

Remark. The last assertion of Theorem 1.4 is a consequence of [4, Theorem 1.4], which is obtained by using the last assertion of [4, Theorem 2.1] in the proof of [4, Theorem 1.4].

Oryan [8], [9], and Zeren [13] considered some power series with rational coefficients and showed that under certain conditions these series take transcendental values at non-zero algebraic number arguments, and they determined the classes of these transcendental values with respect to Mahler's classification. They proved their results by applying Baker's Theorem [2]. (We also refer the reader to Oryan [7] and Zeren [12] for earlier results.)

In the present paper, in Theorem 2.1 and Theorem 2.2, we prove the p-adic analogues of the results of Oryan [9] and Zeren [13], respectively, for non-zero rational number arguments by applying the recent result Theorem 1.4. Our main results are stated and proved in the next section.

2 The main results

By definition, a p-adic Liouville number is a p-adic irrational number ξ such that, for every $w > 1$, there is a rational number x/y such that $|\xi - x/y|_p < |x, y|^{-w}$.

Theorem 2.1. *Let*

$$f(x) = \sum_{k=0}^{\infty} c_k x^{e_k}$$

be a power series in $\mathbb{Q}_p$, *where* $c_k = b_k/a_k$ $(k = 0, 1, 2, \ldots)$ *is a non-zero rational number with* $a_k \geq 1$ *and* $\gcd(a_k, b_k) = 1$, *and* $\{e_k\}_{k=0}^{\infty}$ *is a strictly increasing sequence of non-negative rational integers. Suppose that*

$$|c_k|_p \leq p^{-u_k} \qquad (k = 0, 1, 2, \ldots), \tag{2.1}$$

where u_k $(k = 0, 1, 2, \ldots)$ *is a positive rational integer,*

$$\sigma := \liminf_{k\to\infty} \frac{u_{k+1}}{u_k} > 1, \tag{2.2}$$

$$\lim_{k\to\infty} \frac{u_k}{e_k} = \infty, \tag{2.3}$$

and

$$\lambda := \limsup_{k\to\infty} \frac{\log_p (A_k C_k)}{u_k} < \infty, \tag{2.4}$$

where A_k $(k = 1, 2, 3, \ldots)$ *denotes the least common multiple of the rational integers* $a_0, a_1, \ldots, a_k$ *and* $C_k = \max\{1, |c_0|, \ldots, |c_k|\}$ $(k = 0, 1, 2, \ldots)$. *Then the radius of convergence of the power series* $f(x)$ *is infinite. Let* $\alpha = b/a$ *be a non-zero rational number with* $a \geq 1$ *and* $\gcd(a, b) = 1$. *Assume that*

$$\sigma > 2\lambda. \tag{2.5}$$

Then $f(\alpha)$ *is a p-adic transcendental number. If*

$$\mu := \limsup_{k\to\infty} \frac{u_{k+1}}{u_k} < \infty, \tag{2.6}$$

then $f(\alpha)$ *is either a p-adic S-number or a p-adic T-number. If*

$$\limsup_{k\to\infty} \frac{u_{k+1}}{u_k} = \infty, \tag{2.7}$$

then $f(\alpha)$ *is a p-adic Liouville number. Moreover, if* α *and* c_k, $(k \geq 0)$, *are non-zero integers, then the assumption* (2.5) *can be replaced by the weaker condition*

$$\sigma > \lambda.$$

Proof of Theorem 2.1. We prove Theorem 2.1 by improving the method of the proof of Satz 2 in Zeren [12] via the application of Theorem 1.4 in five steps as follows.

1) By (2.1), we have

$$0 < \sqrt[e_k]{|c_k|_p} \leq p^{-u_k/e_k} \qquad (k = 0, 1, 2, \ldots).$$

By (2.3), the radius of convergence of the power series f is infinite.

2) We define the rational numbers

$$\frac{x_n}{y_n} = \sum_{k=0}^{n} c_k \alpha^{e_k} \qquad (n = 1, 2, 3, \ldots).$$

Then

$$|x_n, y_n| \leq (e_n + 1)|a, b|^{e_n} A_n C_n \leq (2|a, b|)^{e_n} A_n C_n \qquad (n = 1, 2, 3, \ldots). \tag{2.8}$$

We can assume that $\gcd(x_n, y_n) = 1$ $(n = 1, 2, 3, \ldots)$ and shall do so. It follows from (2.4) that

$$A_n C_n < p^{u_n(\lambda + \varepsilon_1)} \tag{2.9}$$

for sufficiently large n, where ε_1 is a positive real number. By (2.3),

$$(2|a, b|)^{e_n} < p^{u_n \varepsilon_2} \tag{2.10}$$

holds for sufficiently large n, where ε_2 is a positive real number. We infer from (2.8), (2.9), and (2.10) that

$$|x_n, y_n| < p^{u_n(\lambda + \varepsilon_1 + \varepsilon_2)} \tag{2.11}$$

for sufficiently large n.

3) Let $|\alpha|_p = p^h$. By (2.3), we have for sufficiently large n

$$1 - \frac{he_n}{u_n} > 1 - \varepsilon_3,$$

where ε_3 is a positive real number with $\varepsilon_3 < 1$. Thus

$$|c_n \alpha^{e_n}|_p \leq p^{-u_n(1 - he_n/u_n)} < p^{-u_n(1 - \varepsilon_3)} \tag{2.12}$$

for sufficiently large n. Then

$$\left| f(\alpha) - \frac{x_n}{y_n} \right|_p \leq \max \left\{ |c_{n+1} \alpha^{e_{n+1}}|_p, |c_{n+2} \alpha^{e_{n+2}}|_p, \ldots \right\} \leq p^{-u_{n+1}(1 - \varepsilon_3)} \tag{2.13}$$

for sufficiently large n because, by (2.2), there exists a positive real number ε_4 with $\sigma - \varepsilon_4 > 1$ such that

$$u_{n+1} > (\sigma - \varepsilon_4)u_n > u_n \tag{2.14}$$

for sufficiently large n. Hence

$$\left| f(\alpha) - \frac{x_n}{y_n} \right|_p < p^{-u_n(\sigma-\varepsilon_4)(1-\varepsilon_3)} \tag{2.15}$$

for sufficiently large n. By (2.11) and (2.15), we get for sufficiently large n

$$\left| f(\alpha) - \frac{x_n}{y_n} \right|_p < |x_n, y_n|^{-(\sigma-\varepsilon_4)(1-\varepsilon_3)/(\lambda+\varepsilon_1+\varepsilon_2)}. \tag{2.16}$$

4) We have $x_n/y_n - x_{n-1}/y_{n-1} = c_n\alpha^{e_n} \neq 0$ $(n = 2, 3, 4, \ldots)$. By (2.12), we obtain for sufficiently large n

$$0 < \left| \frac{x_n}{y_n} - \frac{x_{n-1}}{y_{n-1}} \right|_p < p^{-u_n(1-\varepsilon_3)}.$$

Since

$$\left| \frac{x_n}{y_n} - \frac{x_{n-1}}{y_{n-1}} \right|_p \geq \frac{1}{|x_n y_{n-1} - x_{n-1} y_n|} \geq \frac{1}{2\,|x_n, y_n|\,|x_{n-1}, y_{n-1}|} \qquad (n = 2, 3, 4, \ldots),$$

it follows that

$$\frac{1}{2\,|x_n, y_n|\,|x_{n-1}, y_{n-1}|} < p^{-u_n(1-\varepsilon_3)}$$

for sufficiently large n. Combining this inequality with (2.11) and (2.14), we see that

$$|x_n, y_n| > \frac{1}{2} p^{((\sigma-\varepsilon_4)(1-\varepsilon_3)-(\lambda+\varepsilon_1+\varepsilon_2))u_{n-1}} \tag{2.17}$$

for sufficiently large n. We infer from (2.11) and (2.17) that

$$\frac{|x_n, y_n|}{|x_{n-1}, y_{n-1}|} > \frac{1}{2} p^{((\sigma-\varepsilon_4)(1-\varepsilon_3)-2(\lambda+\varepsilon_1+\varepsilon_2))u_{n-1}}$$

for sufficiently large n. By (2.5), we have $2\lambda < \sigma$. Thus, by the appropriate choices of ε_1, ε_2, ε_3, and ε_4, the first factor in the exponent of p in the inequality above is positive. So

$$|x_n, y_n| > |x_{n-1}, y_{n-1}|$$

holds for sufficiently large n. Then, noting that $\gcd(x_n, y_n) = 1$ $(n = 1, 2, 3, \ldots)$, the rational numbers x_n/y_n are all distinct from each other from some n onward.

5) By (2.5), there exists a positive real number ε such that

$$2 + \varepsilon < \frac{\sigma}{\lambda} - \varepsilon. \tag{2.18}$$

By appropriate choices of ε_1, ε_2, ε_3, and ε_4, we have

$$\frac{(\sigma - \varepsilon_4)(1 - \varepsilon_3)}{\lambda + \varepsilon_1 + \varepsilon_2} > \frac{\sigma}{\lambda} - \varepsilon. \tag{2.19}$$

It follows from (2.16), (2.18), and (2.19) that

$$0 < \left| f(\alpha) - \frac{x_n}{y_n} \right|_p < |x_n, y_n|^{-(2+\varepsilon)} \tag{2.20}$$

for sufficiently large n. Hence, by Theorem 1.2, $f(\alpha)$ is a p-adic transcendental number.

Let (2.6) hold. By (2.5), (2.6), (2.11), (2.17), and the appropriate choices of ε_1, ε_2, ε_3, and ε_4, we see that

$$\limsup_{n\to\infty} \frac{\log |x_{n+1}, y_{n+1}|}{\log |x_n, y_n|} \leq \frac{\mu^2}{\sigma/\lambda - 1} < \mu^2 < \infty. \tag{2.21}$$

It follows from (2.20), (2.21), and the first assertion of Theorem 1.4 that $f(\alpha)$ is either a p-adic S-number or a p-adic T-number.

Let (2.7) hold. By (2.11) and (2.13), we have for sufficiently large n

$$0 < \left| f(\alpha) - \frac{x_n}{y_n} \right|_p < |x_n, y_n|^{-(u_{n+1}/u_n)((1-\varepsilon_3)/(\lambda+\varepsilon_1+\varepsilon_2))}. \tag{2.22}$$

We deduce from (2.7) and (2.22) that $f(\alpha)$ is a p-adic Liouville number.

When α and the c_k are rational integers, we apply the last assertion of Theorem 1.4 instead of the first one. This completes the proof of Theorem 2.1.

Corollary 2.1. *If we take $e_k = k$ $(k = 0, 1, 2, \ldots)$ in Theorem 2.1, then condition (2.3) is implied by condition (2.2). In this case, we obtain the p-adic analogue of the theorem of Oryan [8].*

We establish the following two examples for our result Theorem 2.1.

Example 2.1. *Let α be a non-zero rational number. If we take $c_k = p^{(4^{k+1})}$, $u_k = 3^{k+1}$, $e_k = 2^k$ $(k = 0,1,2,\ldots)$, and $x = \alpha$, then all the conditions of Theorem 2.1 are verified. Hence $\sum_{k=0}^{\infty} p^{(4^{k+1})}\alpha^{(2^k)}$ is either a p-adic S-number or a p-adic T-number.*

Example 2.2. *Let α be a non-zero rational number. Let us take $c_k = p^{\left((k+1)^{k+1}\right)}$, $u_k = (k+1)^{k+1}$, $e_k = (k+1)^2$ $(k = 0,1,2,\ldots)$, and $x = \alpha$ in Theorem 2.1. Then this yields another example for Theorem 2.1. Namely, $\sum_{k=0}^{\infty} p^{\left((k+1)^{k+1}\right)}\alpha^{\left((k+1)^2\right)}$ is a p-adic Liouville number.*

Theorem 2.2. *Let*

$$F(z) = \sum_{h=0}^{\infty} c_h z^h$$

be a power series in $\mathbb{Q}_p$, where $c_h = b_h/a_h$ $(h = 0,1,2,\ldots)$ is a rational number with $a_h \geq 1$ $(h = 0,1,2,\ldots)$, satisfying

$$\begin{cases} c_h = 0, & r_n < h < s_n \quad (n = 1,2,3,\ldots), \\ c_h \neq 0, & h = r_n \quad (n = 1,2,3,\ldots), \\ c_h \neq 0, & h = s_n \quad (n = 0,1,2,\ldots), \end{cases} \tag{2.23}$$

where $\{s_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ are two infinite sequences of non-negative rational integers with

$$0 = s_0 \leq r_1 < s_1 \leq r_2 < s_2 \leq r_3 < s_3 \leq r_4 < s_4 \leq \ldots.$$

Suppose that the radius of convergence R of the series $F(z)$ is positive and finite. Assume that the following conditions hold:

$$\lambda := \limsup_{h\to\infty} \frac{\log A_h}{h} < \infty, \tag{2.24}$$

where A_h $(h = 1,2,3,\ldots)$ denotes the least common multiple of the rational integers $a_0, a_1, \ldots, a_h$,

$$\sigma := \limsup_{h\to\infty} \frac{\log \max\{1, |b_h|\}}{h} < \infty, \tag{2.25}$$

$$\theta := \liminf_{n\to\infty} \frac{s_n}{r_n} > 1, \tag{2.26}$$

and

$$\phi := \limsup_{n\to\infty} \frac{r_n}{s_{n-1}} < \infty. \tag{2.27}$$

Let $\alpha = b/a$ be a rational number with $a \geq 1$ and $\gcd(a,b) = 1$ *such that*

$$0 < |\alpha|_p < R \tag{2.28}$$

and

$$P_k(\alpha) \neq 0$$

from some k onward, where $P_k(z) = \sum_{h=s_k}^{r_{k+1}} c_h z^h$ $(k = 0, 1, 2, \ldots)$. Moreover, assume that

$$\lambda + \sigma + \log |a, b| < \frac{\theta}{2} \log \frac{R}{|\alpha|_p}. \tag{2.29}$$

Then $F(\alpha)$ is a p-adic transcendental number. If

$$\mu := \limsup_{n \to \infty} \frac{s_n}{r_n} < \infty, \tag{2.30}$$

then $F(\alpha)$ is either a p-adic S-number or a p-adic T-number. If

$$\limsup_{n \to \infty} \frac{s_n}{r_n} = \infty, \tag{2.31}$$

then $F(\alpha)$ is a p-adic Liouville number.

Remark. As in Theorem 2.1, in the case where α and c_h, $(h \geq 0)$, are non-zero integers, the assumption (2.29) can be replaced by the weaker assumption

$$\lambda + \sigma + \log |b| < \theta \log \frac{R}{|\alpha|_p}.$$

This follows from the last statement of Theorem 1.4. We omit the details.

Proof of Theorem 2.2. We prove Theorem 2.2 by extending the method of the proof of Satz in Teil II of Zeren [13] via the application of Theorem 1.4 to the p-adic case in three steps as follows.

1) By (2.23), we can write $F(z) = \sum_{k=0}^{\infty} P_k(z)$ for the p-adic numbers z at which $F(z)$ converges. By (2.26), (2.28), and (2.29), we can choose four real numbers λ_1, σ_1, θ_1, and r such that the inequalities

$$\lambda_1 > \lambda, \qquad \sigma_1 > \sigma, \qquad 1 < \theta_1 < \theta, \qquad |\alpha|_p < r < R, \tag{2.32}$$

and

$$\lambda_1 + \sigma_1 + \log |a, b| < \frac{\theta_1}{2} \log \frac{r}{|\alpha|_p} \tag{2.33}$$

hold. It follows from (2.26) and (2.32) that

$$\frac{s_n}{r_n} > \theta_1 \tag{2.34}$$

for sufficiently large n.

We define the rational numbers

$$\frac{x_n}{y_n} = \sum_{k=0}^{n-1} P_k(\alpha) = \sum_{h=s_0}^{r_n} c_h \alpha^h \qquad (n = 1, 2, 3, \dots).$$

Then, using (2.25) and (2.32), we get for sufficiently large n

$$|x_n, y_n| \leq (r_n + 1)|a, b|^{r_n} A_{r_n} \max\{|b_0|, |b_1|, \dots, |b_{r_n}|\} \leq B_{r_n}|a, b|^{r_n}, \tag{2.35}$$

where $B_{r_n} = A_{r_n}(r_n + 1)\exp(\sigma_1 r_n)$. We can assume that $\gcd(x_n, y_n) = 1$ $(n = 1, 2, 3, \dots)$ and shall do so. By (2.24) and (2.32),

$$\limsup_{n\to\infty} \frac{\log B_{r_n}}{r_n} < \lambda_1 + \sigma_1. \tag{2.36}$$

We infer from (2.33), (2.35), and (2.36) that

$$|x_n, y_n| < \exp\left((\lambda_1 + \sigma_1 + \log|a, b|)r_n\right) < \left(\frac{r}{|\alpha|_p}\right)^{\theta_1 r_n/2} \tag{2.37}$$

for sufficiently large n.

2) By the hypothesis of the theorem, we have $x_n/y_n - x_{n-1}/y_{n-1} = P_{n-1}(\alpha) \neq 0$ for sufficiently large n. Then

$$0 < \left|\frac{x_n}{y_n} - \frac{x_{n-1}}{y_{n-1}}\right|_p = |P_{n-1}(\alpha)|_p \leq \max_{h=s_{n-1},\dots,r_n} \left\{|c_h|_p |\alpha|_p^h\right\} \tag{2.38}$$

for sufficiently large n. Since $R = 1/\limsup_{h\to\infty} \sqrt[h]{|c_h|_p}$ and $0 < r < R$, there exists a positive integer h_0 such that

$$|c_h|_p < r^{-h}$$

for $h \geq h_0$. In fact, there is a real number $M \geq 1$ such that

$$|c_h|_p < Mr^{-h} \qquad (h = 1, 2, 3, \dots). \tag{2.39}$$

Hence, by (2.32), (2.38), and (2.39), we have for sufficiently large n

$$0 < \left|\frac{x_n}{y_n} - \frac{x_{n-1}}{y_{n-1}}\right|_p < M\left(\frac{|\alpha|_p}{r}\right)^{s_{n-1}},$$

thus,

$$\frac{1}{2\,|x_n, y_n|\,|x_{n-1}, y_{n-1}|} < M\left(\frac{|\alpha|_p}{r}\right)^{s_{n-1}}$$

holds for sufficiently large n. Combining this inequality with (2.37), we get for sufficiently large n

$$|x_n, y_n| > \frac{1}{2M}\left(\frac{r}{|\alpha|_p}\right)^{s_{n-1}-\theta_1 r_{n-1}/2}. \tag{2.40}$$

It follows from (2.32), (2.34), and (2.40) that

$$|x_n, y_n| > \frac{1}{2M}\left(\frac{r}{|\alpha|_p}\right)^{s_{n-1}/2} \tag{2.41}$$

for sufficiently large n. By (2.26) and (2.32), we can choose a real number θ_2 with

$$1 < \theta_1 < \theta_2 < \theta \tag{2.42}$$

such that

$$\frac{s_n}{r_n} > \theta_2 \tag{2.43}$$

holds for sufficiently large n. Using (2.43) in (2.41), we have for sufficiently large n

$$|x_{n+1}, y_{n+1}| > \frac{1}{2M}\left(\frac{r}{|\alpha|_p}\right)^{\theta_2 r_n/2}. \tag{2.44}$$

We infer from (2.32) and (2.42) that

$$\frac{1}{2M}\left(\frac{r}{|\alpha|_p}\right)^{\theta_2 r_n/2} > \left(\frac{r}{|\alpha|_p}\right)^{\theta_1 r_n/2}$$

for sufficiently large n. Thus, by (2.37) and (2.44),

$$|x_{n+1}, y_{n+1}| > |x_n, y_n|$$

holds for sufficiently large n. Then, noting that $\gcd(x_n, y_n) = 1$ $(n = 1, 2, 3, \ldots)$, the rational numbers x_n/y_n are all distinct from each other from some n onward.

3) By (2.32) and (2.39), we have

$$\left|F(\alpha) - \frac{x_n}{y_n}\right|_p \leq \max\left\{|c_{s_n}|_p|\alpha|_p^{s_n}, |c_{s_n+1}|_p|\alpha|_p^{s_n+1}, \ldots\right\} \leq M\left(\frac{r}{|\alpha|_p}\right)^{-s_n} \tag{2.45}$$

for $n = 1, 2, 3, \ldots$. We deduce from (2.37) and (2.45) that

$$\left| F(\alpha) - \frac{x_n}{y_n} \right|_p < M \, |x_n, y_n|^{-(2/\theta_1)(s_n/r_n)} \tag{2.46}$$

for sufficiently large n. Combining (2.46) with (2.42) and (2.43), we obtain for sufficiently large n

$$0 < \left| F(\alpha) - \frac{x_n}{y_n} \right|_p < |x_n, y_n|^{-\chi}, \tag{2.47}$$

where χ is a real number with $2 < \chi < 2\theta_2/\theta_1$. Hence, by Theorem 1.2, $F(\alpha)$ is a p-adic transcendental number.

Let (2.30) hold. It follows from (2.37) and (2.41) that

$$\frac{\log |x_{n+1}, y_{n+1}|}{\log |x_n, y_n|} < \theta_1 \frac{r_{n+1}}{s_{n-1}} \frac{1}{1 + M_1/s_{n-1}}$$

for sufficiently large n, where $M_1 = \frac{2\log(1/(2M))}{\log(r/|\alpha|_p)}$. We can write

$$\frac{r_{n+1}}{s_{n-1}} = \frac{r_{n+1}}{s_n} \frac{s_n}{r_n} \frac{r_n}{s_{n-1}}.$$

So, by (2.27) and (2.30),

$$\limsup_{n\to\infty} \frac{\log |x_{n+1}, y_{n+1}|}{\log |x_n, y_n|} \le \theta_1 \phi \mu \phi < \infty. \tag{2.48}$$

We deduce from (2.47), (2.48), and Theorem 1.4 that $F(\alpha)$ is either a p-adic S-number or a p-adic T-number.

Let (2.31) hold. In this case, we infer from (2.46) that $F(\alpha)$ is a p-adic Liouville number. This completes the proof of Theorem 2.2.

We establish the following two examples for our result Theorem 2.2.

Example 2.3. *Let $F(z) = \sum_{h=0}^{\infty} c_h z^h$ be a power series in $\mathbb{Q}_p$ with*

$$\begin{cases} c_h = 0, & r_n < h < s_n \quad (n = 1, 2, 3, \ldots), \\ c_h = p^h, & s_n \le h \le r_{n+1} \quad (n = 0, 1, 2, \ldots), \end{cases}$$

where $\{s_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ are two infinite sequences of non-negative rational integers, determined by

$$s_0 = 0, \quad s_n = 5^{n+1} \quad \text{and} \quad r_n = 2 \cdot 5^n \quad (n = 1, 2, 3, \ldots).$$

Then, by Theorem 2.2, $F(p^t)$ is either a p-adic S-number or a p-adic T-number, where t is any positive rational integer.

Example 2.4. *In Example 2.3, if we take the sequences* $\{s_n\}_{n=0}^{\infty}$ *and* $\{r_n\}_{n=1}^{\infty}$ *as*

$$s_0 = 0, \quad s_n = (n+2)^{n+1} \quad \text{and} \quad r_n = 3 \cdot (n+1)^n \quad (n = 1, 2, 3, \ldots),$$

then this yields another example for Theorem 2.2. Namely, $F(p^t)$ *is a* p*-adic Liouville number, where* t *is any positive rational integer.*

Acknowledgements

This work is supported by Scientific Research Projects Coordination Unit of Istanbul University. Project number: FUA-2018-31152.

References

[1] B. Adamczewski, Y. Bugeaud, Mesures de transcendance et aspects quantitatifs de la méthode de Thue-Siegel-Roth-Schmidt, *Proc. London Math. Soc.* **101** (2010), 1-26.

[2] A. Baker, On Mahler's classification of transcendental numbers, *Acta Math.* **111** (1964), 97-120.

[3] Y. Bugeaud, *Approximation by Algebraic Numbers,* Cambridge Tracts in Mathematics, 160 (Cambridge University Press, Cambridge, 2004).

[4] Y. Bugeaud, G. Kekeç, On Mahler's classification of p-adic numbers, *Bull. Aust. Math. Soc.* **98** (2018), 203-211.

[5] K. Mahler, Zur Approximation der Exponentialfunktionen und des Logarithmus. I, II, *J. reine angew. Math.* **166** (1932), 118-150.

[6] K. Mahler, Über eine Klasseneinteilung der p-adischen Zahlen, *Mathematica (Leiden)* **3** (1935), 177-185.

[7] M. H. Oryan, Über gewisse Potenzreihen, die für algebraische Argumente Werte aus der Mahlerschen Unterklassen U_m nehmen, *İstanbul Üniv. Fen Fak. Mecm. Ser. A* **45** (1980), 1-42.

[8] M. H. Oryan, On power series and Mahler's $U-$numbers, *İstanbul Üniv. Fen Fak. Mecm. Ser. A* **47** (1990), 117-125.

[9] M. H. Oryan, On power series and Mahler's $U-$numbers, *Math. Scand.* **65** (1989), 143-151.

[10] D. Ridout, The p-adic generalization of the Thue-Siegel-Roth theorem, *Mathematika* **5** (1958), 40-48.

[11] K. F. Roth, Rational approximations to algebraic numbers, *Mathematika* **2** (1955), 1-20; corrigendum, 168.

[12] B. M. Zeren, Über die Transzendenz der Werte einiger schnell konvergenter Potenzreihen für algebraische Argumente, *İstanbul Tek. Üniv. Bül.* **38** (1985), 473-496.

[13] B. M. Zeren, Über eine Klasse von verallgemeinerten Lückenreihen, deren Werte für algebraische Argumente transzendent, aber keine $U-$Zahlen sind I, *İstanbul Üniv. Fen Fak. Mat. Derg.* **50** (1991), 79-99.

YANN BUGEAUD,
Université de Strasbourg,
CNRS, IRMA UMR 7501,
7, rue René Descartes,
67000 Strasbourg, France.
Email: bugeaud@math.unistra.fr

GÜLCAN KEKEÇ,
Istanbul University,
Faculty of Science,
Department of Mathematics,
34134, Vezneciler-Istanbul, Turkey.
Email: gulkekec@istanbul.edu.tr